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A POSTERIORI ERROR ANALYSIS OF EULER–GALERKIN APPROXIMATIONS TO COUPLED ELLIPTIC–PARABOLIC PROBLEMS *

Alexandre Ern[1] and Sébastien Meunier[2]

Abstract. We analyze Euler–Galerkin approximations (conforming finite elements in space and implicit Euler in time) to coupled PDE systems in which one dependent variable, say u, is governed by an elliptic equation and the other, say p, by a parabolic-like equation. The targeted application is the poroelasticity system within the quasi-static assumption. An abstract setting is proposed to identify a natural energy norm for the PDE system. Two *a posteriori* error analyzes are performed, both yielding reliable upper error bounds in the sense that all the constants are specified. The first analysis hinges directly on the stability of the continuous problem and can be used to estimate the dominant term associated with the p-component in the energy norm. The second analysis is an extension of the elliptic reconstruction technique introduced by Makridakis and Nochetto [*SIAM J. Numer. Anal.*, **41(4)**, 1585–1594, 2003] for linear parabolic problems. It is used here to derive an *a posteriori* error estimate for the u-component in the energy norm that exhibits an optimal convergence rate with respect to mesh size. Numerical results are presented to illustrate the performance of the various estimators.

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1. Introduction

The main motivation for this work is the performance assessment by numerical simulations of underground storage facilities for nuclear waste. The processes involved in near-field models are extremely complex and can involve multi-phase, multicomponent flows through a porous medium subjected to thermal, hydraulic, chemical, and mechanical couplings. Here, we focus on poroelasticity problems involving hydro-mechanical couplings. We consider a linearly elastic and porous medium Ω saturated by a slightly compressible and viscous fluid within the so-called quasi-static assumption in which inertia effects in the elastic structure are negligible. Given a simulation time $T > 0$, the problem consists of finding a displacement field $u : [0,T] \times \Omega \to \mathbb{R}^3$ and a pressure field $p : [0,T] \times \Omega \to \mathbb{R}$ such that

$$-\nabla\cdot\sigma(u) + b\nabla p = f, \qquad \text{in } [0,T] \times \Omega, \tag{1}$$

$$\partial_t(\tfrac{1}{M}p + b\nabla\cdot u) - \nabla\cdot(\kappa\nabla p) = g, \qquad \text{in } [0,T] \times \Omega. \tag{2}$$

Keywords and phrases: Finite element method, *a priori* error analysis, *a posteriori* error analysis, Hydro-Mechanical coupling

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Here, $\sigma(u) = 2\lambda_1\varepsilon(u) + \lambda_2(\nabla\cdot u)I$ is the so-called effective stress tensor, $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^t)$ the linearized strain tensor, λ_1 and λ_2 the Lamé coefficients, I the identity matrix in $\mathbb{R}^3$, b the Biot–Willis coefficient, M the Biot modulus, κ the permeability of the medium, while f and g are given data. The system (1)–(2) is supplemented by initial and boundary conditions discussed below. The poroelasticity system can be traced back to the pioneering work of Terzaghi [22] and Biot [3]. Equations (1)–(2) respectively express the balance of momentum and the conservation of mass. The quasi-static assumption means that the term $\rho\partial_{tt}u$ (where ρ denotes the density of the elastic structure) has been neglected in the momentum balance. The Biot modulus combines compressibility and porosity effects; it is often assumed to be very large when dealing with the so-called Biot's consolidation problem, but this assumption will not be made here. For the sake of simplicity, we will assume that the coefficients b and M are given constants. A mathematical analysis of the system (1)–(2), including existence and uniqueness of strong and weak solutions based on the theory of linear degenerate evolution equations in Hilbert spaces, has been carried out by Showalter [16, 17]. Boundary conditions can be prescribed by considering two partitions of the boundary. The first partition is used for the displacement field (either the displacement itself or a traction force is prescribed), while the other partition is used for the pressure field (either the pressure itself or a flux is prescribed). For the sake of simplicity, we assume here that any portion of the boundary is clamped or drained, i.e. at least a Dirichlet condition is enforced on the displacement or on the pressure everywhere. Furthermore, an initial condition must be enforced on the quantity $\frac{1}{M}p + b\nabla\cdot u$. Although the evolution problem related to (1)–(2) is essentially of parabolic type under minimal smoothness requirements on the data, we refer to it as a coupled elliptic–parabolic problem to stress the fact that Equ. (1) is of elliptic type for the displacement and Equ. (2) is of parabolic type for the pressure.

In the present work, we assume that the data (including boundary and initial conditions) are smooth enough for a strong solution to exist, and we shall mainly be concerned with the *a posteriori* error analysis of Euler–Galerkin approximations to the exact solution obtained by using a backward Euler scheme in time and conforming finite element methods in space. In the sequel, we restrict our attention to the case where the exact solution is smooth up to the initial time, as is customary in the *a posteriori* error analysis of Euler–Galerkin approximations to parabolic problems such as the heat equation. The *a priori* analysis of Euler–Galerkin approximations for Biot's consolidation problem is covered in the work of Murad, Loula, and coworkers [11–13], including the semi-discrete and fully discrete cases and long-time behavior. The problem under scrutiny here is somewhat different since we do not assume that the Biot modulus takes very large values, i.e. we do not discard the pressure time-derivative in (2). As a result, we shall briefly address below the *a priori* error analysis of the Euler–Galerkin approximation to the evolution problem (1)–(2). The natural stability norm associated with this problem controls on the one hand the $L^\infty_t(H^1_x)$-norm (L^∞ in time and H^1 in space) of the displacement and the $L^\infty_t(L^2_x)$-norm of the pressure and on the other hand the $L^2_t(H^1_x)$-norm of the pressure. This key feature implies that error estimates with optimal convergence orders in space require the use of different polynomial degrees in the finite element spaces for the displacement and for the pressure, namely one degree higher for the displacement than for the pressure. Note that the use of different polynomial degrees for approximating the displacement and the pressure is motivated here solely by the derivation of optimal error estimates, as opposed to Biot's consolidation problem where the approximation of the initial data by solving a Stokes problem plays also a role. The technique of Wheeler [23] originally designed to obtain optimal $L^\infty_t(L^2_x)$ *a priori* error estimates for the heat equation can be adapted to fit the present framework to derive optimal error estimates for the displacement field. The same technique has already been used in [11–13] for Biot's consolidation problem.

The *a posteriori* error analysis of evolution problems related to poroelasticity is a much less explored field. The first aim is to derive error estimates that are reliable in the sense that they yield an upper bound for the approximation error (the difference between the exact solution and the discrete solution) that is fully computable in terms of known quantities, i.e. the problem data, universal constants from polynomial interpolation theory, and the discrete solution itself. A further important feature of the error estimates is their so-called optimality in the sense that the estimates are bounded from above by the *a priori* error estimates. A third desirable feature of the estimates is that they provide an efficient tool to adapt the discretization parameters (mesh size and time step). In the present work, we derive fully reliable and partially optimal *a posteriori* error estimates of residual type for the poroelasticity system (1)–(2). The analysis hinges on the stability properties of the continuous problem, so that the natural norms with which to control the approximation error are the $L^\infty_t(H^1_x)$-norm for the displacement and the $L^\infty_t(L^2_x)$- and $L^2_t(H^1_x)$-norms for the pressure. The direct approach yields error estimates that have a form analogous to those derived for the heat equation by Picasso [14], Chen and Feng [4] and Bergam, Bernardi and Mghazli [2]. The error estimate consists of three terms, a time error indicator (evaluated from the pressure differences at two consecutive time steps), a space error indicator (evaluated from the finite element residuals for the displacement and the pressure) and a data oscillation term. Despite these similarities with the heat equation, there are however important differences. Firstly, since stability is achieved by testing the momentum balance against the time-derivative of the displacement, the space error indicator contains a term involving the finite element residuals of the time-derivative of the displacement. This also makes the analysis of time-dependent meshes much more cumbersome than for the heat equation. Therefore, we have chosen to restrict ourselves to fixed meshes and to postpone the study of discretizations with time-dependent meshes to future work. The second important difference is that the estimates derived within the direct approach are not optimal for the approximation error on the displacement when the discretization of the latter involves higher-degree polynomials than for the pressure. To tackle this difficulty, we draw inspiration from the elliptic reconstruction technique introduced for linear parabolic problems by Makridakis and Nochetto [10] and further analyzed by Lakkis and Makridakis [9]. This technique, which can be regarded as the counterpart of the elliptic projection method introduced by Wheeler for the *a priori* error analysis, is designed to obtain optimal *a posteriori* error estimates in the $L^\infty_t(L^2_x)$-norm (and other higher-order norms) for linear parabolic problems. In the present work, we extend this technique to coupled elliptic–parabolic problems such as the poroelasticity equations to derive *a posteriori* error estimates for the displacement that exhibit optimal convergence behavior. Other approaches to derive $L^\infty_t(L^2_x)$ *a posteriori* error estimates for parabolic equations can be found, among others, in the work of Eriksson and Johnson [6, 7] and of Thomée [18] based on duality techniques, the work of Babuška, Feistauer, and Šolín [1] using a double integration in time, and the work of Verfürth [21] for quasi-linear parabolic equations.

This paper is organized as follows. §2 presents the setting under scrutiny in an abstract framework. This setting allows us to pinpoint the mathematical structure of the poroelasticity system that plays a relevant role in the subsequent analysis. §3 is devoted to the *a priori* error analysis. The main result is Theorem 3.1 whose proof relies on the technique of elliptic projection introduced by Wheeler. The arguments are similar to those considered in [11–13] for Biot's consolidation problem, and are briefly presented here for completeness. §4 deals with the *a posteriori* error analysis. The first result, Theorem 4.1, relies on the stability of the continuous problem and is established by proceeding in a way similar to the heat equation. It yields a reliable error estimate for the displacement in the $L^\infty_t(H^1_x)$-norm and for the pressure in the $L^\infty_t(L^2_x)$- and $L^2_t(H^1_x)$-norms. The error upper bound consists of data, time, and space error indicators. The time error indicator is optimal while the space error indicator comprises three terms, two of which yield local lower bounds of the approximation error. The last one exhibits an optimal convergence behavior with respect to mesh size when compared to the pressure error in the $L^2_t(H^1_x)$-norm, but its optimality with respect to the time step cannot be established because it depends on the time-derivative of the discrete displacement field. The second result, Theorem 4.2, yields reliable error estimates for the displacement error in the $L^\infty_t(H^1_x)$-norm (and also for the pressure in the $L^\infty_t(L^2_x)$-norm) and these estimates exhibit the optimal convergence order with respect to mesh size. The proof relies on an adaptation of the technique of elliptic reconstruction introduced recently by Makridakis and Nochetto. §5 contains numerical results illustrating the performance of the various *a posteriori* error estimators. Finally, §6 draws some conclusions.

2. The setting

2.1. **The continuous problem**

Let V_a and V_d be two Hilbert spaces respectively equipped with symmetric, continuous and coercive bilinear forms a and d. The norms induced by these forms are denoted by $\|\cdot\|_a$ and $\|\cdot\|_d$ respectively. Let V'_a (resp.,

V_d') be the dual space of V_a (resp., V_d) with duality product denoted by $\langle\cdot,\cdot\rangle_a$ (resp., $\langle\cdot,\cdot\rangle_d$) and norm $\|\cdot\|_{a'} = \sup_{0\neq v\in V_a} |\langle\cdot,v\rangle_a|/\|v\|_a$ (resp., $\|\cdot\|_{d'} = \sup_{0\neq q\in V_d} |\langle\cdot,q\rangle_d|/\|q\|_d$). Let L_a (resp., L_d) be a Hilbert space equipped with a scalar product $(\cdot,\cdot)_{L_a}$ (resp., $(\cdot,\cdot)_{L_d}$) with dense and continuous injection $V_a \hookrightarrow L_a$ (resp., $V_d \hookrightarrow L_d$). Identifying L_a (resp., L_d) with its dual space, the following injections hold: $V_a \hookrightarrow L_a \equiv (L_a)' \hookrightarrow V_a'$ (resp., $V_d \hookrightarrow L_d \equiv (L_d)' \hookrightarrow V_d'$). Let c be a symmetric, continuous and coercive bilinear form defined over $L_d \times L_d$ inducing a norm $\|\cdot\|_c$ such that for all $q \in V_d$, $\|q\|_c \leq \gamma\|q\|_d$ and $\|q\|_{L_d} \leq \tilde{\gamma}\|q\|_d$. Finally, let b be a continuous bilinear form defined over $V_a \times L_d$ with continuity constant β, i.e., for all $(v,q) \in V_a \times L_d$, $|b(v,q)| \leq \beta\|v\|_a\|q\|_c$.

The elements of the spaces defined above are functions of the space variable x. In the sequel, we shall deal with functions of time and space. The time variable varies over the interval $[0,T]$ for a fixed $T > 0$. Henceforth, $L^2_t(Z)$ denotes the vector space of functions f in space and time such that for a.e. $t \in [0,T]$, $f(t) := f(t,\cdot)$ is in Z (where Z denotes any of the spaces defined above) and $f(t)$ is square integrable over $[0,T]$, i.e. $\int_0^T \|f(s)\|_Z^2 ds < +\infty$. Similarly, $H^1_t(Z)$ denotes the subspace of $L^2_t(Z)$ of functions f with square integrable distributional time-derivative $\partial_t f$ over $[0,T]$. Observe that functions in $H^1_t(Z)$ admit pointwise values in Z for all $t \in [0,T]$.

Given data $f \in H^1_t(V_a')$, $g \in H^1_t(V_d')$, and $p_0 \in V_d$, we seek for the strong solution $(u,p) \in H^1_t(V_a) \times H^1_t(V_d)$ such that for all $t \in [0,T]$,

$$a(u,v) - b(v,p) = \langle f,v\rangle_a, \qquad \forall v \in V_a, \tag{3}$$

$$c(\partial_t p,q) + b(\partial_t u,q) + d(p,q) = \langle g,q\rangle_d, \qquad \forall q \in V_d, \tag{4}$$

completed with the initial condition $p(0) := p_0$. Note that in the present setting, Equ. (3) holds up to $t = 0$, thus uniquely determining the initial value of u in terms of p_0 and $f(0)$. Letting $u_0 := u(0) \in V_a$, the *a priori* bound $\|u_0\|_a \leq \beta\|p_0\|_c + \|f(0)\|_{a'}$ is readily inferred by taking $v := u_0$ in (3).

Our first result is an *a priori* estimate for the strong solution. This will allow us to identify the natural stability norm for problem (3)–(4).

Proposition 2.1. *In the above setting, the following holds for a.e.* $t \in [0,T]$,

$$\tfrac12\|u(t)\|_a^2 + \tfrac12\|p(t)\|_c^2 + \tfrac12\int_0^t \|p(s)\|_d^2 ds \leq 2\left(2\sup_{s\in[0,T]}\|f(s)\|_{a'} + \int_0^T \|\partial_t f(s)\|_{a'} ds\right)^2 + \int_0^T \|g(s)\|_{d'}^2 ds + \|u_0\|_a^2 + \|p_0\|_c^2. \tag{5}$$

Proof. Since (u,p) is a strong solution, for a.e. $t \in [0,T]$, $v = \partial_t u$ is in V_a and $q = p$ is in V_d. Using these test functions in (3)–(4) yields

$$\tfrac12 d_t\|u\|_a^2 + \tfrac12 d_t\|p\|_c^2 + \|p\|_d^2 = \langle f,\partial_t u\rangle_a + \langle g,p\rangle_d.$$

Hence,

$$\tfrac12 d_t\|u\|_a^2 + \tfrac12 d_t\|p\|_c^2 + \tfrac12\|p\|_d^2 \leq \langle f,\partial_t u\rangle_a + \tfrac12\|g\|_{d'}^2.$$

Let $t \in (0,T)$. Integrating the above inequality over $(0,t)$ and integrating by parts in time the term $\langle f,\partial_t u\rangle_a$ yields

$$\tfrac12\|u(t)\|_a^2 + \tfrac12\|p(t)\|_c^2 + \tfrac12\int_0^t \|p(s)\|_d^2 ds \leq \langle f(t),u(t)\rangle_a - \langle f(0),u(0)\rangle_a - \int_0^t \langle\partial_t f(s),u(s)\rangle_a ds + \tfrac12\int_0^t \|g(s)\|_{d'}^2 ds + \tfrac12\|u_0\|_a^2 + \tfrac12\|p_0\|_c^2.$$

As a result, it is inferred that for a.e. $t \in [0,T]$,

$$\tfrac{1}{2}\|u(t)\|_a^2 + \tfrac{1}{2}\|p(t)\|_c^2 + \tfrac{1}{2}\int_0^t \|p(s)\|_d^2 ds \le A\sigma_T(u) + B^2,$$

with $\sigma_T(u) = \sup_{s\in[0,T]} \|u(s)\|_a$,

$$A = 2\sup_{s\in[0,T]} \|f(s)\|_{a'} + \int_0^T \|\partial_t f(s)\|_{a'} ds \qquad \text{and} \qquad B^2 = \tfrac{1}{2}\int_0^T \|g(s)\|_{d'}^2 ds + \tfrac{1}{2}\|u_0\|_a^2 + \tfrac{1}{2}\|p_0\|_c^2.$$

Hence, $\frac{1}{2}\sigma_T(u)^2 \le A\sigma_T(u) + B^2$ so that $\sigma_T(u)^2 \le 4(A^2+B^2)$. This yields

$$\tfrac{1}{2}\|u(t)\|_a^2 + \tfrac{1}{2}\|p(t)\|_c^2 + \tfrac{1}{2}\int_0^t \|p(s)\|_d^2 ds \le A^2 + \tfrac{1}{4}\sigma_T(u)^2 + B^2 \le 2(A^2+B^2),$$

yielding (5). □

An important consequence of Proposition 2.1 is the uniqueness of the strong solution of (3)–(4). In the sequel, we assume the existence of the strong solution.

Remark 2.1. *Proposition 2.1 holds in the slightly more general setting where $g \in L_t^2(V_d')$, $p_0 \in L_d$, and $p \in L_t^2(V_d) \cap H_t^1(L_d)$.*

Application to poroelasticity. The evolution problem (1)–(2) fits the present setting. For the sake of simplicity, we consider homogeneous Dirichlet boundary conditions both for the displacement and the pressure. Then, letting $V_a = [H_0^1(\Omega)]^3$, $L_a = [L^2(\Omega)]^3$, $V_d = H_0^1(\Omega)$ and $L_d = L^2(\Omega)$, we define the bilinear forms

$$a(u,v) = \int_\Omega \sigma(u){:}\varepsilon(v), \qquad b(v,p) = \int_\Omega bp\nabla{\cdot}v, \tag{6}$$

$$c(p,q) = \int_\Omega \tfrac{1}{M}pq, \qquad d(p,q) = \int_\Omega \kappa\nabla p{\cdot}\nabla q. \tag{7}$$

The coercivity of a on $V_a \times V_a$ (resp., d on $V_d \times V_d$) results from Korn's First Inequality (resp., Poincaré's Inequality). The bilinear form b is clearly continuous on $V_a \times L_d$ with $\beta = b(M/E)^{1/2}$ where the Young modulus E is associated with the coercivity of the bilinear form a on V_a. Proposition 2.1 means that the natural stability norm controls the displacement in the $L_t^\infty(H_x^1)$-norm and the pressure in the $L_t^\infty(L_x^2)$- and $L_t^2(H_x^1)$-norms. This stability result hinges on the fact that the momentum balance can be tested by the time-derivative of the displacement.

2.2. The discrete problem

Problem (3)–(4) is approximated by an Euler–Galerkin scheme, namely conforming finite elements in space and an implicit Euler scheme in time. Let $\{V_{ah}\}_{h>0}$ and $\{V_{dh}\}_{h>0}$ be two families of finite-dimensional subspaces of V_a and V_d respectively. The parameter h refers to the size of an underlying mesh family denoted by $\{\mathcal{T}_h\}_{h>0}$. Let $0 = t_0 < t_1 < \ldots < t_N = T$ be a sequence of discrete times and for all $n \in \{1,\ldots,N\}$, set $\tau_n = t_n - t_{n-1}$ and $I_n = (t_{n-1}, t_n)$. Henceforth, a superscript n indicates the value taken by any function of space and time at the discrete time t_n. For instance, $u^n := u(t_n) \in V_a$.

The discrete problem consists of seeking $\{u_h^n\}_{n=1}^N \in [V_{ah}]^N$ and $\{p_h^n\}_{n=1}^N \in [V_{dh}]^N$ such that for all $n \in \{1,\ldots,N\}$,

$$a(u_h^n, v_h) - b(v_h, p_h^n) = (f_h^n, v_h)_{L_a}, \qquad \forall v_h \in V_{ah}, \tag{8}$$

$$c(\delta_t p_h^n, q_h) + b(\delta_t u_h^n, q_h) + d(p_h^n, q_h) = (g_h^n, q_h)_{L_d}, \qquad \forall q_h \in V_{dh}, \tag{9}$$

where $\delta_t p_h^n = \tau_n^{-1}(p_h^n - p_h^{n-1})$ and $\delta_t u_h^n = \tau_n^{-1}(u_h^n - u_h^{n-1})$. Given a pair $(u_{0h}, p_{0h}) \in V_{ah} \times V_{dh}$, the initial condition is $(u_h^0, p_h^0) := (u_{0h}, p_{0h})$. The data $\{f_h^n\}_{n=1}^N \in [V_{ah}]^N \subset [V_a']^N$ and $\{g_h^n\}_{n=1}^N \in [V_{dh}]^N \subset [V_d']^N$ are approximations of $\{f^n\}_{n=1}^N$ and $\{g^n\}_{n=1}^N$ respectively.

Lemma 2.1. *The discrete problem is well–posed.*

Proof. For all $n \in \{1, \ldots, N\}$, Equs. (8)–(9) yield a square linear system for the components of (u_h^n, p_h^n) once bases of V_{ah} and V_{dh} are chosen, so it suffices to prove the uniqueness of the discrete solution. Testing with $v_h = u_h^n - u_h^{n-1}$ and $q_h = \tau_n p_h^n$ and using the fact that $a(x, x-y) = \frac{1}{2}a(x,x) + \frac{1}{2}a(x-y, x-y) - \frac{1}{2}a(y,y)$ owing to the symmetry of the bilinear form a (along with a similar property for the bilinear form c) yields

$$\tfrac{1}{2}\|u_h^n\|_a^2 + \tfrac{1}{2}\|u_h^n - u_h^{n-1}\|_a^2 + \tfrac{1}{2}\|p_h^n\|_c^2 + \tfrac{1}{2}\|p_h^n - p_h^{n-1}\|_c^2 + \tau_n\|p_h^n\|_d^2 = \tfrac{1}{2}\|u_h^{n-1}\|_a^2 + (f_h^n, u_h^n - u_h^{n-1})_{L_a} + \tfrac{1}{2}\|p_h^{n-1}\|_c^2 + \tau_n(g_h^n, p_h^n)_{L_d}.$$

Hence,

$$\tfrac{1}{2}\|u_h^n\|_a^2 + \tfrac{1}{2}\|p_h^n\|_c^2 + \tfrac{1}{2}\tau_n\|p_h^n\|_d^2 \le \tfrac{1}{2}\|u_h^{n-1}\|_a^2 + \tfrac{1}{2}\|f_h^n\|_{a'}^2 + \tfrac{1}{2}\|p_h^{n-1}\|_c^2 + \tfrac{1}{2}\tau_n\|g_h^n\|_{d'}^2.$$

This shows the uniqueness of the solution of the square linear system. □

The proof of Lemma 2.1 hints at how the discrete scheme (8)–(9) could be modified if time-dependent meshes were used. In this case, we work with two families $\{V_{ah}^n\}_{n=0}^N$ and $\{V_{dh}^n\}_{n=0}^N$ of finite-dimensional subspaces such that for all $n \in \{0, \ldots, N\}$, $V_{ah}^n \subset V_a$ and $V_{dh}^n \subset V_d$. The discrete scheme takes the general form (8)–(9) with test functions $v_h \in V_{ah}^n$ and $q_h \in V_{dh}^n$. However, if the expression for the time-derivative of the displacement is kept unchanged, the argument deployed in the above proof breaks down because it is no longer possible to use $v_h = u_h^n - u_h^{n-1}$ as a test function since in general $v_h \notin V_{ah}^n$ (unless the restrictive assumption $V_{ah}^{n-1} \subset V_{ah}^n$ is made for all $n \in \{1, \ldots, N\}$). To circumvent this difficulty, let $\mathfrak{R}_{ah}^{n*} : V_a \to V_{ah}^n$ be the Riesz projection operator defined such that for all $v \in V_a$,

$$a(v - \mathfrak{R}_{ah}^{n*}(v), v_h) = 0, \qquad \forall v_h \in V_{ah}^n, \tag{10}$$

and use $\delta_t u_h^n = \tau_n^{-1}(u_h^n - \mathfrak{R}_{ah}^{n*}(u_h^{n-1}))$ in (9). Then, proceeding as in the proof of Lemma 2.1 with the test functions $v_h = u_h^n - \mathfrak{R}_{ah}^{n*}(u_h^{n-1}) \in V_{ah}^n$ and $q_h = \tau_n p_h^n \in V_{dh}^n$ and observing that $a(u_h^n, v_h) = a(u_h^n, u_h^n - u_h^{n-1})$ since $u_h^n \in V_{ah}^n$, the same stability estimate is recovered.

2.3. Continuous and discrete differential operators

To formulate the *a posteriori* error estimates in the usual form, it is convenient to associate differential operators (in space) with the bilinear forms a, b, c, and d. To this purpose, we define the following continuous operators: $A \in \mathcal{L}(V_a; V_a')$ s.t. $\langle Av, w\rangle_a = -a(v,w)$, $B \in \mathcal{L}(V_a; L_d)$ s.t. $(Bv, q)_{L_d} = b(v,q)$ with adjoint $B^* \in \mathcal{L}(L_d; V_a')$ s.t. $\langle B^* q, v\rangle_a = b(v,q)$, $C \in \mathcal{L}(L_d; L_d)$ s.t. $(Cq, r)_{L_d} = c(q,r)$, and $D \in \mathcal{L}(V_d; V_d')$ s.t. $\langle Dq, r\rangle_d = -d(q,r)$. Observe that C is actually a zero-order operator. With this notation, problem (3)–(4) can be rewritten in the form

$$-Au - B^* p = f, \tag{11}$$

$$C\partial_t p + B\partial_t u - Dp = g, \tag{12}$$

these equalities holding for a.e. $t \in [0,T]$ in V_a' and V_d' respectively. For the poroelasticity system, $Av = \nabla{\cdot}\sigma(v)$, $Bv = b\nabla{\cdot}v$, $B^* q = -b\nabla q$, $Cq = \frac{1}{M}q$, and $Dq = \nabla{\cdot}(\kappa\nabla p)$.

The discrete version of the above operators will also be used: $A_h \in \mathcal{L}(V_{ah}; V_{ah})$ s.t. $(A_h v_h, w_h)_{L_a} = -a(v_h, w_h)$, $B_h \in \mathcal{L}(V_{ah}; V_{dh})$ s.t. $(B_h v_h, q_h)_{L_d} = b(v_h, q_h)$ with adjoint $B_h^* \in \mathcal{L}(V_{dh}; V_{ah})$ s.t. $(B_h^* q_h, v_h)_{L_a} = b(v_h, q_h)$, and $D_h \in \mathcal{L}(V_{dh}; V_{dh})$ s.t. $(D_h q_h, r_h)_{L_d} = -d(q_h, r_h)$. Observe that duality products have been replaced by L_a- and L_d-scalar products. The discrete problem (8)–(9) can be rewritten in the form

$$-A_h u_h^n - B_h^* p_h^n = f_h^n, \tag{13}$$
$$C\delta_t p_h^n + B_h \delta_t u_h^n - D_h p_h^n = g_h^n, \tag{14}$$

these equalities holding for all $n \in \{1,\dots,N\}$ in V_{ah} and V_{dh} respectively. For later use, we let $f_h^0 := -A_h u_{0h} - B_h^* p_{0h}$, so that (13) also holds for $n=0$.

2.4. **The steady problem**

Both the *a priori* and *a posteriori* error analysis of the approximation of the steady version of (3)–(4) using the subspaces $\{V_{ah}\}_{h>0}$ and $\{V_{dh}\}_{h>0}$ will play a role in the error analysis for the time-dependent case. The steady version of (3)–(4) consists of seeking $\overline{u} \in V_a$ and $\overline{p} \in V_d$ such that

$$a(\overline{u}, v) - b(v, \overline{p}) = \langle \overline{f}, v\rangle_a, \qquad \forall v \in V_a, \tag{15}$$
$$d(\overline{p}, q) = \langle \overline{g}, q\rangle_d, \qquad \forall q \in V_d, \tag{16}$$

with data $\overline{f} \in V_a'$ and $\overline{g} \in V_d'$. It is straightforward to verify that the problem (15)–(16) is well–posed owing to its upper triangular structure and the coercivity of the bilinear forms a and d.

The discrete problem consists of seeking $\overline{u}_h \in V_{ah}$ and $\overline{p}_h \in V_{dh}$ such that

$$a(\overline{u}_h, v_h) - b(v_h, \overline{p}_h) = \langle \overline{f}, v_h\rangle_a, \qquad \forall v_h \in V_{ah}, \tag{17}$$
$$d(\overline{p}_h, q_h) = \langle \overline{g}, q_h\rangle_d, \qquad \forall q_h \in V_{dh}. \tag{18}$$

Here, we do not consider an approximation to the data $\overline{f}$ and $\overline{g}$. The discrete problem is conveniently reformulated using a Riesz projection operator $\mathfrak{R}_h : V_a \times V_d \to V_{ah} \times V_{dh}$ such that for all $(v,q) \in V_a \times V_d$, $\mathfrak{R}_h(v,q) := (\mathfrak{R}_{ah}(v,q), \mathfrak{R}_{dh}(q))$ is defined by

$$a(v - \mathfrak{R}_{ah}(v,q), v_h) - b(v_h, q - \mathfrak{R}_{dh}(q)) = 0, \qquad \forall v_h \in V_{ah}, \tag{19}$$
$$d(q - \mathfrak{R}_{dh}(q), q_h) = 0, \qquad \forall q_h \in V_{dh}. \tag{20}$$

It is clear that $(\overline{u}_h, \overline{p}_h)$ solves (17)–(18) if and only if $\overline{u}_h = \mathfrak{R}_{ah}(\overline{u}, \overline{p})$ and $\overline{p}_h = \mathfrak{R}_{dh}(p)$. The approximation properties of the operator $\mathfrak{R}_h$ can be found in [11]. The result is restated here for completeness.

Lemma 2.2. *The following holds for all* $(v,q) \in V_a \times V_d$,

$$\|v - \mathfrak{R}_{ah}(v,q)\|_a \le \inf_{v_h \in V_{ah}} \|v - v_h\|_a + \beta \|q - \mathfrak{R}_{dh}(q)\|_c, \tag{21}$$
$$\|q - \mathfrak{R}_{dh}(q)\|_d = \inf_{q_h \in V_{dh}} \|q - q_h\|_d. \tag{22}$$

Proof. Property (22) is classical. To establish (21), consider the operator $\mathfrak{R}_{ah}^*$ defined by (10) (the upper index n is dropped since meshes are kept fixed in time). Then, observe that since both $\mathfrak{R}_{ah}^*(v)$ and $\mathfrak{R}_{ah}(v,q)$ are in V_{ah},

$$\begin{aligned}
\|\mathfrak{R}_{ah}(v,q) - \mathfrak{R}_{ah}^*(v)\|_a^2 &= a(\mathfrak{R}_{ah}(v,q) - \mathfrak{R}_{ah}^*(v), \mathfrak{R}_{ah}(v,q) - \mathfrak{R}_{ah}^*(v)) \\
&= a(\mathfrak{R}_{ah}(v,q) - v, \mathfrak{R}_{ah}(v,q) - \mathfrak{R}_{ah}^*(v)) + a(v - \mathfrak{R}_{ah}^*(v), \mathfrak{R}_{ah}(v,q) - \mathfrak{R}_{ah}^*(v)) \\
&= -b(\mathfrak{R}_{ah}(v,q) - \mathfrak{R}_{ah}^*(v), q - \mathfrak{R}_{dh}(q)) \\
&\le \beta \|\mathfrak{R}_{ah}(v,q) - \mathfrak{R}_{ah}^*(v)\|_a \|q - \mathfrak{R}_{dh}(q)\|_c.
\end{aligned}$$

Hence, $\|\mathfrak{R}_{ah}(v,q)-\mathfrak{R}^*_{ah}(v)\|_a \le \beta\|q-\mathfrak{R}_{dh}(q)\|_c$ whence it follows by the triangle inequality that

$$\|v-\mathfrak{R}_{ah}(v,q)\|_a \le \|v-\mathfrak{R}^*_{ah}(v)\|_a + \|\mathfrak{R}_{ah}(v,q)-\mathfrak{R}^*_{ah}(v)\|_a \le \|v-\mathfrak{R}^*_{ah}(v)\|_a + \beta\|q-\mathfrak{R}_{dh}(q)\|_c,$$

readily yielding (21). □

To deduce from Lemma 2.2 asymptotic rates of convergence for the approximation error in terms of the parameter h when the exact solution is smooth enough, we introduce the following assumptions.

Hypothesis 2.1. *There exist constants c_1 and c_2, positive real numbers s_a and s_d, and subspaces $W_a \subset V_a$ and $W_d \subset V_d$ respectively equipped with norms $\|\cdot\|_{W_a}$ and $\|\cdot\|_{W_d}$, such that independently of h,*

$$\forall v \in W_a, \qquad \inf_{v_h \in V_{ah}} \|v-v_h\|_a \le c_1 h^{s_a}\|v\|_{W_a}, \tag{23}$$

$$\forall q \in W_d, \qquad \inf_{q_h \in V_{dh}} \|q-q_h\|_d \le c_2 h^{s_d}\|q\|_{W_d}. \tag{24}$$

Hypothesis 2.2. *There exist a constant c_3 and a positive real number δ such that for all $r \in L_d$, the unique solution $\phi \in V_d$ of the dual problem $d(q,\phi)=c(r,q)$ for all $q \in V_d$, is such that there is $\phi_h \in V_{dh}$ satisfying*

$$\|\phi-\phi_h\|_d \le c_3 h^{\delta}\|r\|_c. \tag{25}$$

Hypothesis 2.1 is classical in the context of finite element approximations. It will be used in the *a priori* error analysis. To keep technicalities at a minimum, a version of Hypothesis 2.1 localized to mesh cells is not considered. Hypothesis 2.2 is an elliptic regularity property associated with the bilinear form d on V_d. It is stated here in compact form, the usual statement consisting of assuming that the dual solution ϕ is in a subspace Y_d of V_d where the interpolation estimate (25) holds in the form $\|\phi-\phi_h\|_d \le c_3 h^{\delta}\|\phi\|_{Y_d}$. Hypothesis 2.2 will serve both in the *a priori* and the *a posteriori* error analysis. In the latter case, a sharper statement localized to mesh cells will be introduced in §4.2. For the time being, we will only use the following important consequence of Hypothesis 2.2:

$$\|q-\mathfrak{R}_{dh}(q)\|_c \le c_3 h^{\delta}\|q-\mathfrak{R}_{dh}(q)\|_d. \tag{26}$$

Indeed, letting ϕ be the dual solution associated with $r := q-\mathfrak{R}_{dh}(q)$ and observing that $d(r,\phi_h)=0$ for $\phi_h \in V_{dh}$ yields

$$\|q-\mathfrak{R}_{dh}(q)\|_c^2 = c(r,r) = d(r,\phi) = d(r,\phi-\phi_h) \le \|r\|_d\|\phi-\phi_h\|_d, \tag{27}$$

whence (26) readily follows. An important consequence of (21), (23), and (26) is that for all $(v,q) \in W_a \times W_d$,

$$\|v-\mathfrak{R}_{ah}(v,q)\|_a \le c_1 h^{s_a}\|v\|_{W_a} + \beta c_2 c_3 h^{s_d+\delta}\|q\|_{W_d}. \tag{28}$$

For the purpose of computational efficiency, it is reasonable to balance both sources of error in $\|v-\mathfrak{R}_{ah}(v,q)\|_a$. This motivates the following hypothesis.

Hypothesis 2.3. $s_a = s_d + \delta =: s$.

In the framework of Hypotheses 2.1–2.3, Lemma 2.2 yields for all $(v,q) \in W_a \times W_d$,

$$\|v-\mathfrak{R}_{ah}(v,q)\|_a \le h^s(c_1\|v\|_{W_a} + \beta c_2 c_3\|q\|_{W_d}), \tag{29}$$

$$\|q-\mathfrak{R}_{dh}(q)\|_c \le h^s c_2 c_3\|q\|_{W_d}, \tag{30}$$

$$\|q-\mathfrak{R}_{dh}(q)\|_d \le h^{s-\delta} c_2\|q\|_{W_d}. \tag{31}$$

As a result, whenever the exact solution of the steady problem (15)–(16) is smooth enough, namely $(\overline{u},\overline{p}) \in W_a \times W_d$, the error $\|\overline{p}-\overline{p}_h\|_d$ converges asymptotically as $h^{s-\delta}$ while the error $\|\overline{u}-\overline{u}_h\|_a + \|\overline{p}-\overline{p}_h\|_c$ converges asymptotically as h^s. Since δ is positive, this means that the error $\|\overline{u}-\overline{u}_h\|_a + \|\overline{p}-\overline{p}_h\|_c$ converges at a faster rate than $\|\overline{p}-\overline{p}_h\|_d$. This difference in the convergence rates will be accounted for in the subsequent analysis of the time-dependent problem, the goal being to derive *a priori* and *a posteriori* error bounds that are optimal for $\|p^n - p_h^n\|_d$ on the one hand and for $\|u^n - u_h^n\|_a + \|p^n - p_h^n\|_c$ on the other hand.

Application to poroelasticity. Consider the model problem (1)–(2) with the displacement (resp., the pressure) approximated in space by continuous Lagrange finite elements of degree $k \geq 1$ (resp., $l \geq 1$). Then, Hypothesis (2.1) holds with $s_a := k$, $W_a := [H_0^1(\Omega) \cap H^{k+1}(\mathcal{T}_h)]^3$, $s_d := l$ and $W_d := H_0^1(\Omega) \cap H^{l+1}(\mathcal{T}_h)$, where for $m \geq 0$, $H^m(\mathcal{T}_h)$ denotes the usual broken Sobolev space of order m. Hypothesis 2.2 means that the steady-state version of the pressure equation yields elliptic regularity, namely for all $r \in L^2(\Omega)$, the unique solution $\phi \in H_0^1(\Omega)$ to the dual problem $\int_\Omega \kappa \nabla\phi \cdot \nabla q = \int_\Omega rq$ for all $q \in H_0^1(\Omega)$ is in $H^2(\Omega)$. Then, (25) holds with $\delta := 1$. As a result, Hypothesis 2.3 implies

$$k = l + 1, \tag{32}$$

i.e. the polynomial interpolation for the displacement is one degree higher than that for the pressure. The most common choice in practice is $k := 2$ and $l := 1$, i.e. continuous piecewise quadratics are used to approximate the displacement and continuous piecewise linears are used to approximate the pressure.

3. *A priori* error analysis

The *a priori* error analysis is performed under the assumption that the exact solution is smooth, namely

$$u \in C_t^1(W_a) \cap C_t^2(V_a), \qquad p \in C_t^1(W_d) \cap C_t^2(L_d). \tag{33}$$

For all $n \in \{1, \dots, N\}$, define

$$C_1^n(u,p) = 2\gamma^2 c_2^2 c_3^2 \sup_{s\in I_n} \|\partial_t p(s)\|_{W_d}^2 + 2\beta^2\gamma^2 (c_1 \sup_{s\in I_n} \|\partial_t u(s)\|_{W_a} + \beta c_2 c_3 \sup_{s\in I_n} \|\partial_t p(s)\|_{W_d})^2, \tag{34}$$

$$C_2^n(u,p) = \tfrac{1}{2}\gamma^2 \sup_{s\in I_n} \|\partial_{tt}^2 p(s)\|_c^2 + \tfrac{1}{2}\beta^2\gamma^2 \sup_{s\in I_n} \|\partial_{tt}^2 u(s)\|_a^2, \tag{35}$$

$$C^n(f,g) = \tfrac{1}{2}\|f^n - f_h^n\|_{a'}^2 + \tau_n \|g^n - g_h^n\|_{d'}^2. \tag{36}$$

Moreover, it is assumed that the initial data u_{0h} and p_{0h} are chosen such that

$$\|u_0 - u_{0h}\|_a \leq c_4 h^s \|u_0\|_{W_a} \qquad \text{and} \qquad \|p_0 - p_{0h}\|_c \leq c_5 h^s \|p_0\|_{W_d}, \tag{37}$$

and we define

$$C(u_0,p_0) = (c_1 + c_4)^2 \|u_0\|_{W_a}^2 + (\beta^2 c_2^2 c_3^2 + \tfrac{1}{2}(c_2 c_3 + c_5)^2)\|p_0\|_{W_d}^2. \tag{38}$$

One possible choice is $u_{0h} = \mathfrak{R}_{ah}(u_0,p_0)$ and $p_{0h} = \mathfrak{R}_{dh}(p_0)$, in which case we can take $C(u_0,p_0) = 0$.

Theorem 3.1. *In the above framework, the following holds for all $n \in \{1, \dots, N\}$,*

$$\begin{aligned} \tfrac{1}{4}\|u^n - u_h^n\|_a^2 + \tfrac{1}{4}\|p^n - p_h^n\|_c^2 \leq h^{2s} C(u_0,p_0) + \sum_{m=1}^n C^m(f,g) + \sum_{m=1}^n [\tau_m h^{2s} C_1^m(u,p) + \tau_m^3 C_2^m(u,p)] \\ + h^{2s}(c_1^2 \|u^n\|_{W_a}^2 + (\beta^2 + \tfrac{1}{2}) c_2^2 c_3^2 \|p^n\|_{W_d}^2), \end{aligned} \tag{39}$$

and

$$\begin{aligned} \sum_{m=1}^n \tfrac{1}{8}\tau_m \|p^m - p_h^m\|_d^2 \leq h^{2s} C(u_0,p_0) + \sum_{m=1}^n C^m(f,g) + \sum_{m=1}^n [\tau_m h^{2s} C_1^m(u,p) + \tau_m^3 C_2^m(u,p)] \\ + \sum_{m=1}^n \tfrac{1}{4}\tau_m h^{2(s-\delta)} c_2^2 \|p^m\|_{W_d}^2. \end{aligned} \tag{40}$$

Proof. (i) For all $n \in \{1, \dots, N\}$, let us first estimate the quantities

$$\eta_{ah}^n = \mathfrak{R}_{ah}(u^n, p^n) - u_h^n \qquad \text{and} \qquad \eta_{dh}^n = \mathfrak{R}_{dh}(p^n) - p_h^n.$$

Observe that

$$\begin{aligned} a(\eta_{ah}^n, v_h) - b(v_h, \eta_{dh}^n) &= \langle f^n - f_h^n, v_h \rangle_a, && \forall v_h \in V_{ah}, \\ c(\eta_{dh}^n - \eta_{dh}^{n-1}, q_h) + b(\eta_{ah}^n - \eta_{ah}^{n-1}, q_h) + \tau_n d(\eta_{dh}^n, q_h) &= \tau_n \langle g^n - g_h^n, q_h \rangle_d + c(\theta_{dh}^n, q_h) + b(\theta_{ah}^n, q_h), && \forall q_h \in V_{dh}, \end{aligned}$$

where $\theta_{dh}^n = \mathfrak{R}_{dh}(p^n) - \mathfrak{R}_{dh}(p^{n-1}) - \tau_n \partial_t p^n$ and $\theta_{ah}^n = \mathfrak{R}_{ah}(u^n, p^n) - \mathfrak{R}_{ah}(u^{n-1}, p^{n-1}) - \tau_n \partial_t u^n$. Testing with $v_h := \eta_{ah}^n - \eta_{ah}^{n-1} \in V_{ah}$ and $q_h := \eta_{dh}^n \in V_{dh}$ yields after some straightforward algebra

$$\begin{aligned} \tfrac{1}{2}\|\eta_{ah}^n\|_a^2 + \tfrac{1}{2}\|\eta_{ah}^n - \eta_{ah}^{n-1}\|_a^2 + \tfrac{1}{2}\|\eta_{dh}^n\|_c^2 + \tfrac{1}{2}\|\eta_{dh}^n - \eta_{dh}^{n-1}\|_c^2 + \tau_n \|\eta_{dh}^n\|_d^2 = \tfrac{1}{2}\|\eta_{ah}^{n-1}\|_a^2 + \tfrac{1}{2}\|\eta_{dh}^{n-1}\|_c^2 \\ + \langle f^n - f_h^n, \eta_{ah}^n - \eta_{ah}^{n-1} \rangle_a + \tau_n \langle g^n - g_h^n, \eta_{dh}^n \rangle_d + c(\theta_{dh}^n, \eta_{dh}^n) + b(\theta_{ah}^n, \eta_{dh}^n). \end{aligned}$$

Hence,

$$\tfrac{1}{2}\|\eta_{ah}^n\|_a^2 + \tfrac{1}{2}\|\eta_{dh}^n\|_c^2 + \tfrac{1}{4}\tau_n \|\eta_{dh}^n\|_d^2 \le \tfrac{1}{2}\|\eta_{ah}^{n-1}\|_a^2 + \tfrac{1}{2}\|\eta_{dh}^{n-1}\|_c^2 + C^n(f, g) + \tau_n^{-1}\gamma^2 \|\theta_{dh}^n\|_c^2 + \tau_n^{-1}\beta^2\gamma^2 \|\theta_{ah}^n\|_a^2.$$

(ii) Let us now estimate the quantities θ_{dh}^n and θ_{ah}^n. Observe that

$$\theta_{dh}^n = -\int_{I_n} [\partial_t p(s) - \mathfrak{R}_{dh}(\partial_t p(s))] ds - \int_{I_n} (s - t_{n-1}) \partial_{tt}^2 p(s) ds.$$

Hence, owing to the regularity assumptions on the exact solution and estimate (30),

$$\|\theta_{dh}^n\|_c \le \tau_n h^s c_2 c_3 \sup_{s \in I_n} \|\partial_t p(s)\|_{W_d} + \tfrac{1}{2}\tau_n^2 \sup_{s \in I_n} \|\partial_{tt}^2 p(s)\|_c.$$

Similarly, using (29),

$$\|\theta_{ah}^n\|_a \le \tau_n h^s (c_1 \sup_{s \in I_n} \|\partial_t u(s)\|_{W_a} + \beta c_2 c_3 \sup_{s \in I_n} \|\partial_t p(s)\|_{W_d}) + \tfrac{1}{2}\tau_n^2 \sup_{s \in I_n} \|\partial_{tt}^2 u(s)\|_a.$$

Therefore,

$$\tau_n^{-1}\gamma^2 \|\theta_{dh}^n\|_c^2 + \tau_n^{-1}\beta^2\gamma^2 \|\theta_{ah}^n\|_a^2 \le \tau_n h^{2s} C_1^n(u, p) + \tau_n^3 C_2^n(u, p).$$

Summing up the above estimates leads to

$$\tfrac{1}{2}\|\eta_{ah}^n\|_a^2 + \tfrac{1}{2}\|\eta_{dh}^n\|_c^2 + \sum_{m=1}^n \tfrac{1}{4}\tau_m \|\eta_{dh}^m\|_d^2 \le \tfrac{1}{2}\|\eta_{ah}^0\|_a^2 + \tfrac{1}{2}\|\eta_{dh}^0\|_c^2 + \sum_{m=1}^n [C^m(f, g) + \tau_m h^{2s} C_1^m(u, p) + \tau_m^3 C_2^m(u, p)].$$

(iii) We now estimate the initial errors η_{ah}^0 and η_{dh}^0. In the case where $u_{0h} = \mathfrak{R}_{ah}(u_0, p_0)$ and $p_{0h} = \mathfrak{R}_{dh}(p_0)$, it is clear that $\eta_{ah}^0 = 0$ and $\eta_{dh}^0 = 0$. In the general case, use the triangle inequality to infer

$$\|\eta_{dh}^0\|_c = \|\mathfrak{R}_{dh}(p_0) - p_{0h}\|_c \le \|\mathfrak{R}_{dh}(p_0) - p_0\|_c + \|p_0 - p_{0h}\|_c \le h^s (c_2 c_3 + c_5) \|p_0\|_{W_d}.$$

Similarly,

$$\|\eta_{ah}^0\|_a \le \|\mathfrak{R}_{ah}(u_0, p_0) - u_0\|_a + \|u_0 - u_{0h}\|_a \le h^s ((c_1 + c_4) \|u_0\|_{W_a} + \beta c_2 c_3 \|p_0\|_{W_d}).$$

Hence,

$$\tfrac{1}{2}\|\eta_{ah}^n\|_a^2 + \tfrac{1}{2}\|\eta_{dh}^n\|_c^2 + \sum_{m=1}^{n} \tfrac{1}{4}\tau_m\|\eta_{dh}^m\|_d^2 \le h^{2s}C(u_0,p_0) + \sum_{m=1}^{n}[C^m(f,g) + \tau_m h^{2s}C_1^m(u,p) + \tau_m^3 C_2^m(u,p)].$$

(iv) The conclusion readily results from the triangle inequality and estimates (29)–(30)–(31). □

Theorem 3.1 shows that whenever the exact solution is smooth enough and up to data approximation errors that can be made small enough, $\|u^n - u_h^n\|_a + \|p^n - p_h^n\|_c$ converges to order s in space and first-order in time, while $(\sum_{m=1}^n \tau_m\|p^m - p_h^m\|_d^2)^{1/2}$ converges to order $(s-\delta)$ in space and first-order in time.

Application to poroelasticity. When continuous piecewise quadratics (resp., linears) are used to approximate the displacement (resp., the pressure), $\|u^n - u_h^n\|_{H^1} + \|p^n - p_h^n\|_{L^2}$ converges to second-order in space and first-order in time, while $(\sum_{m=1}^n \tau_m\|p^m - p_h^m\|_{H^1}^2)^{1/2}$ converges to first-order in space and in time.

4. *A posteriori* error analysis

This section is devoted to the *a posteriori* error analysis for the discrete scheme (8)–(9). Two estimates are derived. The first directly relies on the stability of the continuous problem and yields an error estimate that is suitable to measure the error in p in the $L_t^2(V_d)$-norm. The second is based on an adaption of the elliptic reconstruction technique of Makridakis and Nochetto and yields an error estimate that is suitable to measure the error in u in the $L_t^\infty(V_a)$-norm and the error in p in the $L_t^\infty(L_d)$-norm. As is customary in *a posteriori* error analysis, we assume in this section that the data (f,g) are in $L_t^2(L_a)\times L_t^2(L_d)$.

4.1. The direct approach

The *a posteriori* error analysis relies on the stability of the continuous problem. Therefore, we rewrite the discrete scheme as equations holding a.e. in $(0,T)$ rather than at the discrete times $\{t_n\}_{n=1}^N$. To this purpose, let $u_{h\tau}$ (resp., $p_{h\tau}$) be the continuous and piecewise affine function in time such that for all $n\in\{0,\dots,N\}$, $u_{h\tau}(t_n) = u_h^n$ (resp., $p_{h\tau}(t_n) = p_h^n$). Observe that $\partial_t u_{h\tau}$ and $\partial_t p_{h\tau}$ are defined a.e. in $(0,T)$. Similarly, let $f_{h\tau}$ be the continuous and piecewise affine function in time such that for all $n\in\{0,\dots,N\}$, $f_{h\tau}(t_n) = f_h^n$. We will also need to consider piecewise constant functions in time, namely $\pi^0 p_{h\tau}$ (resp., $\pi^0 g_{h\tau}$) equal to p_h^n (resp., g_h^n) on I_n for all $n\in\{1,\dots,N\}$. With the above notation, the discrete scheme (8)–(9) yields a.e. in $(0,T)$,

$$a(u_{h\tau}, v_h) - b(v_h, p_{h\tau}) = (f_{h\tau}, v_h)_{L_a}, \qquad \forall v_h \in V_{ah}, \tag{41}$$

$$c(\partial_t p_{h\tau}, q_h) + b(\partial_t u_{h\tau}, q_h) + d(\pi^0 p_{h\tau}, q_h) = (\pi^0 g_{h\tau}, q_h)_{L_d}, \qquad \forall q_h \in V_{dh}. \tag{42}$$

To formulate the *a posteriori* error estimate, it is convenient to introduce the Galerkin residual $\mathfrak{G}_a$ (resp., $\mathfrak{G}_d$) which is a continuous and piecewise affine function in time with values in V_a' (resp., piecewise constant function in time with values in V_d') such that a.e. in $(0,T)$,

$$\langle \mathfrak{G}_a, v\rangle_a = (f_{h\tau}, v)_{L_a} - a(u_{h\tau}, v) + b(v, p_{h\tau}), \qquad \forall v\in V_a, \tag{43}$$

$$\langle \mathfrak{G}_d, q\rangle_d = (\pi^0 g_{h\tau}, q)_{L_d} - c(\partial_t p_{h\tau}, q) - b(\partial_t u_{h\tau}, q) - d(\pi^0 p_{h\tau}, q), \qquad \forall q \in V_d. \tag{44}$$

Define the data, space, and time error estimators

$$\mathcal{E}(f,g) = \int_0^T \|(g-\pi^0 g_{h\tau})(s)\|_{d'}^2 ds + \left(2\sup_{s\in[0,T]} \|(f-f_{h\tau})(s)\|_{a'} + \int_0^T \|\partial_t(f-f_{h\tau})(s)\|_{a'} ds\right)^2, \tag{45}$$

$$\mathcal{E}_{\mathrm{dat}} = \|u_0-u_{0h}\|_a^2 + \|p_0-p_{0h}\|_c^2 + 4\mathcal{E}(f,g), \tag{46}$$

$$\mathcal{E}_{\mathrm{spc}} = \sum_{m=1}^N 4\tau_m \|\mathfrak{G}_d^m\|_{d'}^2 + 4\left(2\sup_{0\le m\le N} \|\mathfrak{G}_a^m\|_{a'} + \sum_{m=1}^N \|\mathfrak{G}_a^m-\mathfrak{G}_a^{m-1}\|_{a'}\right)^2, \tag{47}$$

$$\mathcal{E}_{\mathrm{tim}} = \sum_{m=1}^N \tfrac{1}{3}\tau_m \|p_h^m - p_h^{m-1}\|_d^2. \tag{48}$$

Theorem 4.1. *For all $n\in\{1,\dots,N\}$,*

$$\tfrac12\|u^n-u_h^n\|_a^2 + \tfrac12\|p^n-p_h^n\|_c^2 + \int_0^{t_n} \tfrac14\|(p-p_{h\tau})(s)\|_d^2 ds + \int_0^{t_n} \tfrac12\|(p-\pi^0 p_{h\tau})(s)\|_d^2 ds \le \mathcal{E}_{\mathrm{dat}} + \mathcal{E}_{\mathrm{spc}} + \mathcal{E}_{\mathrm{tim}}. \tag{49}$$

Proof. Let $\xi = u-u_{h\tau}$, $\zeta = p-p_{h\tau}$, and $\zeta^* = p-\pi^0 p_{h\tau}$. Observe that a.e. in $(0,T)$,

$$\begin{aligned} a(\xi,v) - b(v,\zeta) &= \langle f - f_{h\tau} + \mathfrak{G}_a, v\rangle_a, && \forall v\in V_a,\\ c(\partial_t\zeta,q) + b(\partial_t\xi,q) + d(\zeta^*,q) &= \langle g-\pi^0 g_{h\tau} + \mathfrak{G}_d, q\rangle_d, && \forall q\in V_d. \end{aligned}$$

Testing for a.e. $t\in(0,T)$ with $v := \partial_t\xi$ and $q=\zeta$ yields

$$\tfrac12 d_t\|\xi\|_a^2 + \tfrac12 d_t\|\zeta\|_c^2 + \tfrac12\|\zeta\|_d^2 + \tfrac12\|\zeta^*\|_d^2 = \langle f-f_{h\tau}+\mathfrak{G}_a, \partial_t\xi\rangle_a + \langle g-\pi^0 g_{h\tau}+\mathfrak{G}_d,\zeta\rangle_d + \tfrac12\|p_{h\tau}-\pi^0 p_{h\tau}\|_d^2,$$

where we have used the fact that owing to the symmetry of d,

$$d(\zeta,\zeta^*) = \tfrac12 d(\zeta,\zeta) + \tfrac12 d(\zeta^*,\zeta^*) - \tfrac12 d(\zeta-\zeta^*,\zeta-\zeta^*).$$

Since $f-f_{h\tau}+\mathfrak{G}_a \in H_t^1(V_a')$ and $g-\pi^0 g_{h\tau}+\mathfrak{G}_d\in L_t^2(V_d')$, we can proceed as in the proof of Proposition 2.1 and infer for all $n\in\{1,\dots,N\}$, the bound (details are skipped for brevity)

$$\begin{aligned} \tfrac12\|u^n-u_h^n\|_a^2 + \tfrac12\|p^n-p_h^n\|_c^2 + \int_0^{t_n} \tfrac14\|(p-p_{h\tau})(s)\|_d^2 ds + \int_0^{t_n} \tfrac12\|(p-\pi^0 p_{h\tau})(s)\|_d^2 ds \le \mathcal{E}_{\mathrm{dat}} \\ + \int_0^T 4\|\mathfrak{G}_d(s)\|_{d'}^2 ds + 4\left(2\sup_{s\in[0,T]}\|\mathfrak{G}_a(s)\|_{a'} + \int_0^T \|\partial_t\mathfrak{G}_a(s)\|_{a'}\right)^2 ds + \int_0^T \|(p_{h\tau}-\pi^0 p_{h\tau})(s)\|_d^2 ds. \end{aligned}$$

The second and third terms in the right-hand side yield $\mathcal{E}_{\mathrm{spc}}$ and the last term yields $\mathcal{E}_{\mathrm{tim}}$. The final bound (49) results from the fact that $\mathfrak{G}_a$ is piecewise affine, $\mathfrak{G}_d$ is piecewise constant, and that for $s\in I_m$, $(p_{h\tau}-\pi^0 p_{h\tau})(s) = \tau_m^{-1}(s-t_m)(p_h^m - p_h^{m-1})$. ☐

Remark 4.1. *The convergence rate of the upper bound in* (49) *is optimal for $\int_0^{t_n}\frac12\|(p-p_{h\tau})(s)\|_d^2 ds$, but not for $\frac12\|u^n-u_h^n\|_a^2 + \frac12\|p^n-p_h^n\|_c^2$. The last term in the left-hand side of* (49) *is kept only to prove an optimality property of the time error estimator; see Proposition* 4.2.

To localize the space error estimator $\mathcal{E}_{\mathrm{spc}}$, we need to introduce some additional notation and assumptions. We assume that the various bilinear forms in the model problem (3)–(4) can be localized as follows: for all

$(v,q) \in V_a \times V_d$ and for all $(\xi,\zeta) \in V_a \times V_d$ such that either $\xi \in V_{ah}$ or $A\xi \in L_a$ and either $\zeta \in V_{dh}$ or $D\zeta \in L_d$,

$$a(\xi,v) = \sum_{T\in\mathcal{T}_h} [-(A\xi,v)_{L_a(T)} + (J_a\xi,v)_{L_a(\partial T)}], \tag{50}$$

$$d(\zeta,q) = \sum_{T\in\mathcal{T}_h} [-(D\zeta,q)_{L_d(T)} + (J_d\zeta,q)_{L_d(\partial T)}]. \tag{51}$$

Here, for a mesh cell $T \in \mathcal{T}_h$, $L_a(T)$ and $L_d(T)$ are local versions of L_a and L_d respectively, $(\cdot,\cdot)_{L_a(\partial T)}$ and $(\cdot,\cdot)_{L_d(\partial T)}$ are scalar products for functions defined on the boundary ∂T of T and J_a and J_d are suitable (jump) operators such that $J_a\xi = 0$ if $A\xi \in L_a$ and $J_d\zeta = 0$ if $D\zeta \in L_d$. In addition, for all $v \in V_a$ and for all $\zeta \in L_d$ such that either $\zeta \in V_{dh}$ or $B^*\zeta \in L_a$,

$$b(v,\zeta) = \sum_{T\in\mathcal{T}_h} [(v,B^*\zeta)_{L_a(T)} + (J_b\zeta,v)_{L_a(\partial T)}], \tag{52}$$

with $J_b\zeta = 0$ if $B^*\zeta \in L_a$. Moreover, the bilinear forms b and c are also localized as follows: for all $(v,q,r) \in V_a \times L_d \times L_d$,

$$b(v,q) = \sum_{T\in\mathcal{T}_h} (Bv,q)_{L_d(T)} \qquad \text{and} \qquad c(q,r) = \sum_{T\in\mathcal{T}_h} (Cq,r)_{L_d(T)}. \tag{53}$$

Finally, we assume that there exist two (Clément-type) interpolation operators $i_{ah} : V_a \to V_{ah}$ and $i_{dh} : V_d \to V_{dh}$ such that for all $(v,q) \in V_a \times V_d$,

$$\sum_{T\in\mathcal{T}_h} [h_T^{-2}\|v - i_{ah}(v)\|^2_{L_a(T)} + h_T^{-1}\|v - i_{ah}(v)\|^2_{L_a(\partial T)}] \le c_6\|v\|_a^2, \tag{54}$$

$$\sum_{T\in\mathcal{T}_h} [h_T^{-2}\|q - i_{dh}(q)\|^2_{L_d(T)} + h_T^{-1}\|q - i_{dh}(q)\|^2_{L_d(\partial T)}] \le c_7\|q\|_d^2, \tag{55}$$

where for all $T \in \mathcal{T}_h$, h_T denotes the diameter of T. This assumption is classical in the context of *a posteriori* error estimation; see, e.g. [8, 20]. In the context of poroelasticity where $V_a = [H_0^1(\Omega)]^3$ and $V_d = H_0^1(\Omega)$, the usual Clément [5] interpolation operator (modified to account for homogeneous Dirichlet boundary conditions) or the Scott–Zhang [15] interpolation operator can be used.

We define the following elementwise and jump residuals for all $m \in \{1,\dots,N\}$,

$$R^m_{uh} = f^m_h + Au^m_h + B^*p^m_h, \qquad J^m_{uh} = J_a u^m_h - J_b p^m_h, \tag{56}$$

$$R^m_{ph} = g^m_h - C\delta_t p^m_h - B\delta_t u^m_h + Dp^m_h, \qquad J^m_{ph} = J_d p^m_h. \tag{57}$$

For $m = 0$, R^0_{uh}, J^0_{uh}, and J^0_{ph} are defined similarly, while we set $R^0_{ph} = (D - D_h)p_{0h}$ for later use in §4.2. We also define for all $m \in \{0,\dots,N\}$, the space error estimators

$$\widehat{\mathcal{E}}^m_u = \sum_{T\in\mathcal{T}_h} [h_T^2\|R^m_{uh}\|^2_{L_a(T)} + h_T\|J^m_{uh}\|^2_{L_a(\partial T)}], \tag{58}$$

$$\widehat{\mathcal{E}}^m_{p,\alpha} = \sum_{T\in\mathcal{T}_h} h_T^{2\alpha}[h_T^2\|R^m_{ph}\|^2_{L_d(T)} + h_T\|J^m_{ph}\|^2_{L_d(\partial T)}], \tag{59}$$

for a real parameter $\alpha \geq 0$. We will also use the time-incremental version of these estimators, namely for $m \in \{1, \ldots, N\}$,

$$\widehat{\mathcal{E}}_u^m(\delta_t) = \sum_{T \in \mathcal{T}_h} \tau_m^2 [h_T^2 \|\delta_t R_{uh}^m\|_{L_a(T)}^2 + h_T \|\delta_t J_{uh}^m\|_{L_a(\partial T)}^2], \tag{60}$$

$$\widehat{\mathcal{E}}_{p,\alpha}^m(\delta_t) = \sum_{T \in \mathcal{T}_h} h_T^{2\alpha} \tau_m^2 [h_T^2 \|\delta_t R_{ph}^m\|_{L_d(T)}^2 + h_T \|\delta_t J_{ph}^m\|_{L_d(\partial T)}^2], \tag{61}$$

where $\delta_t R_{uh}^m = \tau_m^{-1}(R_{uh}^m - R_{uh}^{m-1})$, $\delta_t J_{uh}^m = \tau_m^{-1}(J_{uh}^m - J_{uh}^{m-1})$, and so on.

Proposition 4.1. *In the above framework, for all $m \in \{0, \ldots, N\}$,*

$$\|\mathfrak{G}_a^m\|_{\prime_a}^2 \leq c_6 \widehat{\mathcal{E}}_u^m, \tag{62}$$

and for all $m \in \{1, \ldots, N\}$,

$$\|\mathfrak{G}_a^m - \mathfrak{G}_a^{m-1}\|_{\prime_a}^2 \leq c_6 \widehat{\mathcal{E}}_u^m(\delta_t), \tag{63}$$

$$\|\mathfrak{G}_d^m\|_{\prime_d}^2 \leq c_7 \widehat{\mathcal{E}}_{p,0}^m. \tag{64}$$

Hence,

$$\mathcal{E}_{\text{spc}} \leq \sum_{m=1}^N 4c_7 \tau_m \widehat{\mathcal{E}}_{p,0}^m + 16c_6 \sup_{0 \leq m \leq N} \widehat{\mathcal{E}}_u^m + 8c_6 \left(\sum_{m=1}^N (\widehat{\mathcal{E}}_u^m(\delta_t))^{1/2} \right)^2. \tag{65}$$

Proof. The proof is only sketched since it uses classical techniques of *a posteriori* error analysis. Observe that

$$\|\mathfrak{G}_a^m\|_{\prime_a} = \sup_{0 \neq v \in V_a} \frac{\langle \mathfrak{G}_a^m, v \rangle_a}{\|v\|_a} = \sup_{0 \neq v \in V_a} \frac{\langle \mathfrak{G}_a^m, v - i_{ah}(v) \rangle_a}{\|v\|_a},$$

since (41) and (43) imply that $\langle \mathfrak{G}_a^m, v_h \rangle_a = 0$ for all $v_h \in V_{ah}$. Then, use definitions (50) and (52), estimate (54) and a Cauchy–Schwarz inequality to infer (62). The proof of (63) and (64) is similar. Finally, (65) results from the definition of $\mathcal{E}_{\text{spc}}$. □

We now investigate the optimality of the space and time error estimators derived above, namely that these quantities yield lower bounds for the errors estimated in Equ. (49). The optimality of the time error estimator is straightforward.

Proposition 4.2. *The following holds*

$$\mathcal{E}_{\text{tim}} \leq 2 \int_0^T \|(p - p_{h\tau})(s)\|_d^2 ds + 2 \int_0^T \|(p - \pi^0 p_{h\tau})(s)\|_d^2 ds. \tag{66}$$

Proof. Observe that $\mathcal{E}_{\text{tim}} = \int_0^T \|(p_{h\tau} - \pi^0 p_{h\tau})(s)\|_d^2 ds$ and use the triangle inequality. □

To investigate the space error estimator, we consider the three terms in the right-hand side of (65). The first two can be bounded using the technique of bubble functions introduced by Verfürth [19, 20]. Since this technique is well-known, the arguments below are only briefly sketched. To alleviate the notation, we denote by $x \lesssim y$ the inequality $x \leq cy$ with positive and mesh-independent constant c. We assume that the bilinear forms a, b, c, and d can be localized elementwise as $a(v,w) = \sum_{T \in \mathcal{T}_h} a_T(v,w)$ and so on; this induces a localization of the norms in the form $\|v\|_a^2 = \sum_{T \in \mathcal{T}_h} \|v\|_{a,T}^2$ with $\|v\|_{a,T}^2 = a_T(v,v)$ and so on. For all $T \in \mathcal{T}_h$, the elementwise residuals R_{uh}^m and R_{ph}^m are in finite-dimensional spaces $P_a(T)$ and $P_d(T)$, respectively, while for a face F in the mesh, the restrictions of the jump residuals J_{uh}^m and J_{ph}^m to F are in finite-dimensional spaces $P_a(F)$ and $P_d(F)$, respectively. For all $T \in \mathcal{T}_h$, we assume that there is a bubble function $\nu_{a,T} \in V_a$ (resp., $\nu_{d,T} \in V_d$) with support localized in T such that for all $v \in P_a(T)$, $\|v\|^2_{L_a(T)} \lesssim (\nu_{a,T}v, v)_{L_a(T)}$ and $\|\nu_{a,T}v\|_{L_a(T)} + h_T\|\nu_{a,T}v\|_{a,T} \leq \|v\|_{L_a(T)}$ (resp., for all $q \in P_d(T)$, $\|q\|^2_{L_d(T)} \lesssim (\nu_{d,T}q, q)_{L_d(T)}$ and $\|\nu_{d,T}q\|_{L_d(T)} + h_T\|\nu_{d,T}q\|_{d,T} \leq \|q\|_{L_d(T)}$). Moreover, for a face $F \subset \partial T$, we assume that there is a bubble function $\nu_{a,F} \in V_a$ (resp., $\nu_{d,F} \in V_d$) with support localized in F and a lifting operator $\pi_{a,F}$ (resp., $\pi_{d,F}$) from F to Δ_F (the set formed by the one or two elements of $\mathcal{T}_h$ to which F belongs) such that for all $v \in P_a(F)$, $\|v\|^2_{L_a(F)} \lesssim (\nu_{a,F}v, v)_{L_a(F)}$ and $h_T^{-1/2}\|\pi_{a,F}(\nu_{a,F}v)\|_{L_a(\Delta_F)} + h_T^{1/2}\|\pi_{a,F}(\nu_{a,F}v)\|_{a,\Delta_F} \lesssim \|v\|_{L_a(F)}$ (resp., for all $q \in P_d(F)$, $\|q\|^2_{L_d(F)} \lesssim (\nu_{d,F}q, q)_{L_d(F)}$ and $h_T^{-1/2}\|\pi_{d,F}(\nu_{d,F}q)\|_{L_d(\Delta_F)} + h_T^{1/2}\|\pi_{d,F}(\nu_{d,F}q)\|_{d,\Delta_F} \lesssim \|q\|_{L_d(F)}$).

Proposition 4.3. *For all $T \in \mathcal{T}_h$, let Δ_T denote the set of mesh cells that share at least a face with T. Then, the following holds*

$$h_T^2\|R^m_{uh}\|^2_{L_a(T)} + h_T\|J^m_{uh}\|^2_{L_a(\partial T)} \lesssim \sum_{T' \in \Delta_T} [h_T^2\|f^m - f^m_h\|^2_{L_a(T')} + \|u^m - u^m_h\|^2_{a,T'} + \|p^m - p^m_h\|^2_{c,T'}], \quad (67)$$

$$\begin{aligned}\tau_m[h_T^2\|R^m_{ph}\|^2_{L_d(T)} + h_T\|J^m_{ph}\|^2_{L_d(\partial T)}] \lesssim \sum_{T' \in \Delta_T} [\int_{I_m} & h_T^2\|(g - \pi^0 g_{h\tau})(s)\|^2_{L_d(T')} ds \quad (68)\\ &+ \|u^m - u^m_h - u^{m-1} + u^{m-1}_h\|^2_{a,T'} + \|p^m - p^m_h - p^{m-1} + p^{m-1}_h\|^2_{c,T'}\\ &+ \int_{I_m} \|(p - \pi^0 p_{h\tau})(s)\|^2_{d,T'} ds].\end{aligned}$$

Proof. Letting $\psi_T = \nu_{a,T}R^m_{uh}$ yields

$$\begin{aligned}\|R^m_{uh}\|^2_{L_a(T)} \lesssim (R^m_{uh}, \psi_T)_{L_a(T)} &= (f^m_h - f^m, \psi_T)_{L_a(T)} + (A(u^m_h - u^m) + B^*(p^m_h - p^m), \psi_T)_{L_a(T)}\\ &= (f^m_h - f^m, \psi_T)_{L_a(T)} + a_T(u^m - u^m_h, \psi_T) - b_T(\psi_T, p^m - p^m_h),\end{aligned}$$

so that

$$h_T\|R^m_{uh}\|_{L_a(T)} \lesssim h_T\|f^m_h - f^m\|_{L_a(T)} + \|u^m - u^m_h\|_{a,T} + \|p^m - p^m_h\|_{c,T}.$$

Similarly, using $\psi_F = \nu_{a,F}J^m_{uh}$ yields

$$\begin{aligned}(J^m_{uh}, \psi_F)_{L_a(F)} = \sum_{T' \in \Delta_F} [&a_{T'}(u^m - u^m_h, \pi_{a,F}\psi_F) - b_{T'}(\pi_{a,F}\psi_F, p^m - p^m_h)\\ &- (R^m_{uh}, \pi_{a,F}\psi_F)_{L_a(T')} + (f^m_h - f^m, \pi_{a,F}\psi_F)_{L_a(T')}],\end{aligned}$$

whence (67) is readily deduced. The proof of (68) is similar. □

Estimates (67)–(68) show that the first two terms bounding $\mathcal{E}_{\mathrm{spc}}$ in (65) yield local lower bounds for the approximation error. A similar result cannot be inferred for the last term in (65) because this term involves time-derivatives of the discrete u-component whereas the error in the u-component is only bounded in the $L^\infty_t(V_a)$-norm. Hence, this term behaves optimally with respect to the mesh size, but not necessarily with respect to the time step if the error in the u-component is not smooth in time. This observation is directly linked with the fact that the analysis relies on the natural stability norm for the continuous problem which provides a control on the u-component only in the $L^\infty_t(V_a)$-norm. Further theoretical work is needed to tackle this issue. In the present work, we will only verify in the numerical experiments presented in §5 that if the exact solution has a smooth behavior in time, this last term also converges optimally.

4.2. **Estimates using elliptic reconstruction**

For all $n \in \{0, \ldots, N\}$, we define the elliptic reconstruction of $(u_h^n, p_h^n) \in V_{ah} \times V_{dh}$ as the functions $(U^n, P^n) \in V_a \times V_d$ such that

$$a(U^n, v) - b(v, P^n) = a(u_h^n, P_{ah}v) - b(P_{ah}v, p_h^n), \qquad \forall v \in V_a, \tag{69}$$
$$d(P^n, q) = d(p_h^n, P_{dh}q) - b(\delta_t u_h^n, q - P_{dh}q), \qquad \forall q \in V_d, \tag{70}$$

where P_{ah} (resp., P_{dh}) denotes the L_a-orthogonal projection from V_a onto V_{ah} (resp., the L_d-orthogonal projection from V_d onto V_{dh}). Henceforth, we use the convention that $\delta_t u_h^0 = 0$ and $\delta_t p_h^0 = 0$. Observe that

$$AU^n + B^* P^n = A_h u_h^n + B_h^* p_h^n, \tag{71}$$
$$DP^n = D_h p_h^n + (B - B_h)\delta_t u_h^n. \tag{72}$$

Indeed, for all $q \in V_d$,

$$\begin{aligned}\langle DP^n, q\rangle_d &= -d(P^n, q) = -d(p_h^n, P_{dh}q) + b(\delta_t u_h^n, q - P_{dh}q)\\ &= (D_h p_h^n, P_{dh}q)_{L_d} + ((B - B_h)\delta_t u_h^n, q)_{L_d} = (D_h p_h^n + (B - B_h)\delta_t u_h^n, q)_{L_d},\end{aligned} \tag{73}$$

since $D_h p_h^n \in V_{dh}$. Equ. (71) is proved similarly.

The key idea to estimate the errors $\|u^n - u_h^n\|_a$ and $\|p^n - p_h^n\|_c$ is to consider the decompositions

$$u^n - u_h^n = \omega_u^n - \rho_u^n, \qquad \omega_u^n = U^n - u_h^n, \qquad \rho_u^n = U^n - u^n, \tag{74}$$
$$p^n - p_h^n = \omega_p^n - \rho_p^n, \qquad \omega_p^n = P^n - p_h^n, \qquad \rho_p^n = P^n - p^n. \tag{75}$$

The quantities ω_u^n and ω_p^n can be bounded by *a posteriori* error estimates for the steady problem, while the quantities ρ_u^n and ρ_p^n can be estimated in terms of ω_u^n and ω_p^n and other computable quantities. The analysis requires a refinement of Hypothesis 2.2 by localizing the approximation property to mesh cells. In the context of poroelasticity, this assumption means that the steady-state version of the pressure equation yields elliptic regularity and that the finite element space V_{dh} satisfies the usual approximation properties.

Hypothesis 4.1. *There exist a constant c_8 and a positive real number δ such that for all $r \in L_d$, the unique solution $\phi \in V_d$ of the dual problem $d(q, \phi) = c(r, q)$ for all $q \in V_d$, is such that there is $\phi_h \in V_{dh}$ satisfying*

$$\sum_{T \in \mathcal{T}_h} h_T^{-2\delta} [h_T^{-2} \|\phi - \phi_h\|^2_{L_d(T)} + h_T^{-1} \|\phi - \phi_h\|^2_{L_d(\partial T)}] \le c_8 \|r\|_c^2. \tag{76}$$

We first estimate the quantities ω_u^n and ω_p^n.

Lemma 4.1. *In the above framework, the following holds for all $n \in \{0, \ldots, N\}$,*

$$\|\omega_u^n\|_a^2 \le 2c_6 \widehat{\mathcal{E}}_u^n + 2\beta^2 c_8 \widehat{\mathcal{E}}_{p,\delta}^n, \tag{77}$$
$$\|\omega_p^n\|_c^2 \le c_8 \widehat{\mathcal{E}}_{p,\delta}^n. \tag{78}$$

Proof. (i) Estimate of $\|\omega_p^n\|_c$. Let ϕ be the dual solution associated with the data $r := \omega_p^n$ in Hypothesis 4.1. Then,

$$\|\omega_p^n\|_c^2 = c(r, r) = d(r, \phi) = d(r, \phi - \phi_h),$$

since owing to (70), $d(r, \phi_h) = d(P^n - p_h^n, \phi_h) = 0$ for $\phi_h \in V_{dh}$. Using (51) and (76) leads to

$$\|\omega_p^n\|_c^2 \le c_8 \sum_{T \in \mathcal{T}_h} h_T^{2\delta} [h_T^2 \|D\omega_p^n\|^2_{L_d(T)} + h_T \|J_d \omega_p^n\|^2_{L_d(\partial T)}].$$

Using (14) and (72) yields for $n \geq 1$,

$$D\omega_p^n = DP^n - Dp_h^n = D_h p_h^n + (B - B_h)\delta_t u_h^n - Dp_h^n = -R_{ph}^n,$$

and this relation also holds for $n = 0$ by definition of R_{ph}^0. In addition, for all $n \geq 0$, $J_d P^n = 0$ since $DP^n \in V_{dh} \subset L_d$. As a result,

$$\|\omega_p^n\|_c^2 \leq c_8 \widehat{\mathcal{E}}_{p,\delta}^n.$$

(ii) Estimate of $\|\omega_u^n\|_a$. Observe that

$$\begin{aligned}\|\omega_u^n\|_a = \sup_{0\neq v\in V_a} \frac{a(\omega_u^n, v)}{\|v\|_a} &= \sup_{0\neq v\in V_a} \left(\frac{a(\omega_u^n, v) - b(v, \omega_p^n)}{\|v\|_a}\right) + \beta\|\omega_p^n\|_c \\ &= \sup_{0\neq v\in V_a} \left(\frac{a(\omega_u^n, v - i_{ah}(v)) - b(v - i_{ah}(v), \omega_p^n)}{\|v\|_a}\right) + \beta\|\omega_p^n\|_c.\end{aligned}$$

owing to (69) since $i_{ah}(v) \in V_{ah}$. Using (50), (52), and (54) leads to

$$\|\omega_u^n\|_a^2 \leq 2c_6 \sum_{T\in\mathcal{T}_h} [h_T^2 \|A\omega_u^n + B^*\omega_p^n\|_{L_a(T)}^2 + h_T \|J_a\omega_u^n - J_b\omega_p^n\|_{L_a(\partial T)}^2] + 2\beta^2 \|\omega_p^n\|_c^2.$$

Owing to (13) and (71), $AU^n + B^*P^n = A_h u_h^n + B_h^* p_h^n = -f_h^n$ so that $A\omega_u^n + B^*\omega_p^n = AU^n + B^*P^n - Au_h^n - B^*p_h^n = -R_{uh}^n$. Moreover, $J_a\omega_u^n - J_b\omega_p^n = -J_{uh}^n$ since $AU^n + B^*P^n \in L_a$. This yields (77). □

We now turn our attention to the quantities ρ_u^n and ρ_p^n. Define the data, space, and time error estimators

$$\widehat{\mathcal{E}}_{\text{dat}} = \|U^0 - u_0\|_a^2 + \|P^0 - p_0\|_c^2 + 2\mathcal{E}(f, g), \tag{79}$$

$$\widehat{\mathcal{E}}_{\text{spc}} = \sum_{m=1}^{N} \tau_m^{-1} [2\gamma^2 c_8 (1 + 2\beta^4) \widehat{\mathcal{E}}_{p,\delta}^m(\delta_t) + 4\beta^2\gamma^2 c_6 \widehat{\mathcal{E}}_u^m(\delta_t)], \tag{80}$$

$$\widehat{\mathcal{E}}_{\text{tim}} = \tfrac{2}{3}\tilde{\gamma}^2 \tau_1 \|C\delta_t p_h^1 + B\delta_t u_h^1 - D_h p_{0h} - g_h^1\|_{L_d}^2 + \sum_{m=2}^{N} \tfrac{2}{3}\tilde{\gamma}^2 \tau_m^3 \|\delta_t (C\delta_t p_h^m + B\delta_t u_h^m - g_h^m)\|_{L_d}^2. \tag{81}$$

Lemma 4.2. *The following holds for all $n \in \{1, \ldots, N\}$,*

$$\tfrac{1}{2}\|\rho_u^n\|_a^2 + \tfrac{1}{2}\|\rho_p^n\|_c^2 \leq \widehat{\mathcal{E}}_{\text{dat}} + \widehat{\mathcal{E}}_{\text{spc}} + \widehat{\mathcal{E}}_{\text{tim}}. \tag{82}$$

Proof. The estimates for ρ_u^n and ρ_p^n rely on the stability properties of the continuous problem. Thus, it is again convenient to handle equations holding a.e. in $[0, T]$ rather than at the discrete times $\{t_n\}_{n=0}^N$. Let U_τ (resp., P_τ) be the continuous and piecewise affine function in time such that for all $n \in \{0, \ldots, N\}$, $U_\tau^n = U^n$ (resp., $P_\tau^n = P^n$). Let $\omega_{u\tau}$ and $\omega_{p\tau}$ be constructed in a similar way from $\{\omega_u^n\}_{n=0}^N$ and $\{\omega_p^n\}_{n=0}^N$. Define $\rho_{u\tau} = U_\tau - u$ and $\rho_{p\tau} = P_\tau - p$. Observe that for a.e. $t \in [0, T]$ and for all $v \in V_a$,

$$\begin{aligned}a(\rho_{u\tau}, v) - b(v, \rho_{p\tau}) &= a(U_\tau, v) - b(v, P_\tau) - (f, v)_{L_a} \\ &= a(u_{h\tau}, P_{ah}v) - b(P_{ah}v, p_{h\tau}) - (f, v)_{L_a} \\ &= (f_{h\tau}, P_{ah}v)_{L_a} - (f, v)_{L_a} = (f_{h\tau} - f, v)_{L_a},\end{aligned}$$

while for all $q \in V_d$,

$$\begin{aligned}c(\partial_t\rho_{p\tau}, q) + b(\partial_t\rho_{u\tau}, q) + d(\rho_{p\tau}, q) &= c(\partial_t P_\tau, q) + b(\partial_t U_\tau, q) + d(P_\tau, q) - (g, q)_{L_d} \\ &= c(\partial_t\omega_{p\tau}, q) + b(\partial_t\omega_{u\tau}, q) + (D\pi^0 P_\tau, q)_{L_d} + d(P_\tau, q) + (\pi^0 g_{h\tau} - g, q)_{L_d} \\ &= c(\partial_t\omega_{p\tau}, q) + b(\partial_t\omega_{u\tau}, q) + (D(\pi^0 P_\tau - P_\tau), q)_{L_d} + (\pi^0 g_{h\tau} - g, q)_{L_d},\end{aligned}$$

where $\pi^0 P_\tau$ is the piecewise constant function in time equal to P^n on I_n for all $n \in \{1, \dots, N\}$. Indeed, on each time interval I_n,

$$\begin{aligned} c(\partial_t p_{h\tau}, q) + b(\partial_t u_{h\tau}, q) &= (C\delta_t p_h^n + B\delta_t u_h^n, q)_{L_d} \\ &= (g_h^n + D_h p_h^n - B_h\delta_t u_h^n + B\delta_t u_h^n, q)_{L_d} = (g_h^n + DP^n, q)_{L_d}. \end{aligned}$$

Testing the above equations with $v := \partial_t \rho_{u\tau}$ and $q := \rho_{p\tau}$ yields

$$\begin{aligned} \tfrac{1}{2}d_t\|\rho_{u\tau}\|_a^2 + \tfrac{1}{2}d_t\|\rho_{p\tau}\|_c^2 + \|\rho_{p\tau}\|_d^2 = (f_{h\tau} - f, \partial_t\rho_{u\tau})_{L_a} + c(\partial_t\omega_{p\tau}, \rho_{p\tau}) + b(\partial_t\omega_{u\tau}, \rho_{p\tau}) \\ + (D(\pi^0 P_\tau - P_\tau), \rho_{p\tau})_{L_d} + (\pi^0 g_{h\tau} - g, \rho_{p\tau})_{L_d}, \end{aligned}$$

so that

$$\begin{aligned} \tfrac{1}{2}d_t\|\rho_{u\tau}\|_a^2 + \tfrac{1}{2}d_t\|\rho_{p\tau}\|_c^2 \le (f_{h\tau} - f, \partial_t\rho_{u\tau})_{L_a} + \gamma^2\|\partial_t\omega_{p\tau}\|_c^2 + \beta^2\gamma^2\|\partial_t\omega_{u\tau}\|_a^2 \\ + \tilde{\gamma}^2\|D(\pi^0 P_\tau - P_\tau)\|_{L_d}^2 + \|\pi^0 g_{h\tau} - g\|_{\tilde{d}}^2. \end{aligned}$$

Proceeding as usual yields for all $n \in \{1, \dots, N\}$ the bound (details are skipped for brevity)

$$\tfrac{1}{2}\|\rho_u^n\|_a^2 + \tfrac{1}{2}\|\rho_p^n\|_c^2 \le \widehat{\mathcal{E}}_{\rm dat} + \int_0^T 2\gamma^2\|\partial_t\omega_{p\tau}(s)\|_c^2 ds + \int_0^T 2\beta^2\gamma^2\|\partial_t\omega_{u\tau}(s)\|_a^2 ds + \int_0^T 2\tilde{\gamma}^2\|D(\pi^0 P_\tau - P_\tau)(s)\|_{L_d}^2 ds.$$

The second and third terms in the upper bound yield the space error estimator $\widehat{\mathcal{E}}_{\rm spc}$ owing to the fact that $\omega_{p\tau}$ and $\omega_{u\tau}$ are piecewise affine in time so that

$$\int_0^T \|\partial_t\omega_{p\tau}(s)\|_c^2 ds = \sum_{m=1}^N \tau_m^{-1}\|\omega_p^m - \omega_p^{m-1}\|_c^2.$$

A similar bound holds for $\omega_{u\tau}$. By linearity and proceeding as in the proof of Lemma 4.1 yields

$$\|\omega_u^m - \omega_u^{m-1}\|_a^2 \le 2c_6\widehat{\mathcal{E}}_u^m(\delta_t) + 2\beta^2 c_8\widehat{\mathcal{E}}_{p,\delta}^m(\delta_t) \qquad \text{and} \qquad \|\omega_p^m - \omega_p^{m-1}\|_c^2 \le c_8\widehat{\mathcal{E}}_{p,\delta}^m(\delta_t).$$

Finally, the last term in the upper bound yields the time error estimator $\widehat{\mathcal{E}}_{\rm tim}$ since

$$\int_0^T \|D(\pi^0 P_\tau - P_\tau)(s)\|_{L_d}^2 ds = \sum_{m=1}^N \tfrac{1}{3}\tau_m\|D(P^m - P^{m-1})\|_{L_d}^2,$$

and for all $m \ge 0$, $DP^m = Dp_h^m - R_{ph}^m$. □

Theorem 4.2. *For all $n \in \{1, \dots, N\}$,*

$$\tfrac{1}{4}\|u^n - u_h^n\|_a^2 + \tfrac{1}{4}\|p^n - p_h^n\|_c^2 \le \widehat{\mathcal{E}}_{\rm dat} + \widehat{\mathcal{E}}_{\rm spc} + \widehat{\mathcal{E}}_{\rm tim} + c_6\widehat{\mathcal{E}}_u^n + c_8(\tfrac{1}{2} + \beta^2)\widehat{\mathcal{E}}_{p,\delta}^n. \tag{83}$$

Proof. Use Lemma 4.1, Lemma 4.2, and the triangle inequality. □

We will not attempt here to prove lower error bounds for the above error estimators; this goes beyond the present scope. We will verify numerically in the following section that these estimators yield the expected, optimal, order of convergence with respect to mesh size.

h_0^{-1}	$\|u-u_{h\tau}\|_a$		$\|p-p_{h\tau}\|_c$		$(\int_0^T \|p-p_{h\tau}\|_d^2)^{1/2}$		$(\int_0^T \|p-\pi^0 p_{h\tau}\|_d^2)^{1/2}$	
4	8.12e-3	–	5.66e-3	–	2.75e-2	–	2.75e-2	–
8	2.15e-3	1.92	1.49e-3	1.92	1.45e-2	0.92	1.45e-2	0.92
16	5.34e-4	2.01	3.73e-4	2.00	7.33e-3	0.98	7.33e-3	0.98
32	1.32e-4	2.02	9.21e-5	2.01	3.68e-3	0.99	3.68e-3	0.99

Table 1. Errors at final time and convergence rates under space refinement ; $T = 0.1$, $\tau = 2.50\text{e-}4$

τ	$\|u-u_{h\tau}\|_a$		$\|p-p_{h\tau}\|_c$		$(\int_0^T \|p-p_{h\tau}\|_d^2)^{1/2}$		$(\int_0^T \|p-\pi^0 p_{h\tau}\|_d^2)^{1/2}$	
0.25	5.10e-3	–	5.64e-3	–	1.68e-2	–	1.81e-2	–
0.2	4.13e-3	0.94	4.55e-3	0.96	1.39e-2	0.84	1.48e-2	0.93
0.1	2.12e-3	0.96	2.31e-3	0.98	7.58e-3	0.87	7.74e-3	0.93
0.05	1.07e-3	0.99	1.16e-3	0.99	4.24e-3	0.83	4.26e-3	0.86

Table 2. Errors at final time and convergence rates under time refinement ; T=1, $h_0 = 1/128$

5. Numerical results

We consider the following analytical solution of (1)-(2) on the domain $\Omega = (0,1)\times(0,1)$,

$$u(t,x,y) = -\frac{\exp(-At)}{2\pi}\begin{bmatrix}\cos(\pi x)\sin(\pi y)\\ \sin(\pi x)\cos(\pi y)\end{bmatrix}, \quad p(t,x,y) = \exp(-At)\sin(\pi x)\sin(\pi y),$$

with $A = \frac{2\pi^2\kappa}{b+\frac{1}{M}}$, $\kappa = 0.05$, $b = 0.75$, $\frac{1}{M} = \frac{3}{28}$. The Lamé coefficients are $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \frac{1}{8}$, yielding a Poisson ratio $\nu = 0.4$ and a Young modulus $E = \frac{7}{20}$. Convergence rates in space are evaluated on a series of uniformly refined structured triangulations based on a boundary mesh step h_0.

Tables 1 and 2 present the convergence results (under space and time refinement respectively) for the approximation errors measured in various norms. All the convergence rates match those predicted by the *a priori* error analysis. An important observation is that in all cases, the total error is dominated by the $L^2_t(H^1_x)$-error on the p-component.

To assess the *a posteriori* error estimate obtained with the direct approach, see (49), we evaluate the quantities

$$\eta_1 = \left(\sum_{m=1}^{N} \tau_m \widehat{\mathcal{E}}^m_{p,0}\right)^{\frac{1}{2}}, \quad \eta_2 = \sup_{0\le m\le N} (\widehat{\mathcal{E}}^m_u)^{\frac{1}{2}}, \quad \eta_3 = \sum_{m=1}^{N} (\widehat{\mathcal{E}}^m_u(\delta_t))^{\frac{1}{2}}, \quad \eta_4 = \left(\sum_{m=1}^{N} \tau_m \|p_h^m - p_h^{m-1}\|_d^2\right)^{\frac{1}{2}}, \tag{84}$$

as well as the efficiency indices

$$\mathcal{I}_{\text{eff}} = \frac{\eta_1+\eta_2+\eta_3+\eta_4}{(\int_0^T [\|p-p_{h\tau}\|_d^2 + \|p-\pi^0 p_{h\tau}\|_d^2])^{1/2}}, \qquad \mathcal{I}^*_{\text{eff}} = \frac{\eta_1+\eta_2+\eta_3+\eta_4}{\|u^N - u^N_{h\tau}\|_a + \|p^N - p^N_{h\tau}\|_c}. \tag{85}$$

Recall that η_1, η_2, and η_3 are associated with the space error estimator and that η_4 is associated with the time error estimator. For brevity, we concentrate here on these two estimators. Tables 3 and 4 present the results obtained under space and time refinement, respectively. All the observed convergence rates match the theoretical predictions. Moreover, the efficiency index $\mathcal{I}_{\text{eff}}$ takes values between 2 and 3, indicating that the present estimators behave quite satisfactorily to control the pressure error in the $L^2_t(H^1_x)$-norm. As expected, the situation is quite different if one attempts to control the displacement error in the $L^\infty_t(H^1_x)$-norm. As reflected by the efficiency index $\mathcal{I}^*_{\text{eff}}$ which increases as the mesh is refined, this latter error converges faster to zero than the estimator derived using the direct approach.

h_0^{-1}	η_1		η_2		η_3		η_4		$\mathcal{I}_{\text{eff}}$	$\mathcal{I}^*_{\text{eff}}$
4	6.34e-2	–	9.53e-2	–	8.17e-3	–	1.45e-3	–	3.13	11.43
8	3.33e-2	0.93	2.57e-2	1.89	2.75e-3	1.57	7.67e-4	0.92	2.18	15.33
16	1.71e-2	0.96	6.63e-3	1.96	7.13e-4	1.94	3.89e-4	0.98	1.70	23.81
32	8.62e-3	0.98	1.68e-3	1.98	1.80e-4	1.99	1.96e-4	0.99	1.45	41.19

TABLE 3. *A posteriori* error estimates using the direct approach and convergence rates under space refinement ; $T = 0.1$, $\tau = 2.50\text{e-}4$

τ	η_1	η_2	η_3	η_4		$\mathcal{I}_{\text{eff}}$	$\mathcal{I}^*_{\text{eff}}$
0.25	4.33e-3	1.07e-4	1.15e-4	4.70e-2	–	1.48	4.80
0.2	4.37e-3	1.02e-4	1.04e-4	3.85e-2	0.90	1.51	4.95
0.1	4.45e-3	9.93e-5	8.25e-5	2.01e-2	0.93	1.62	5.58
0.05	4.49e-3	1.02e-4	7.38e-5	1.03e-2	0.96	1.77	6.69

TABLE 4. *A posteriori* error estimates using the direct approach and convergence rates under time refinement ; $T = 1$, $h_0 = 1/128$

h_0^{-1}	η_5		η_6	η_7		η_8		$\mathcal{J}_{\text{eff}}$
4	1.34e-0	–	1.14e-2	1.55e-2	–	2.18e-2	–	169.43
8	2.01e-1	2.74	7.82e-3	5.70e-3	1.44	5.83e-3	1.90	120.26
16	4.36e-2	2.21	7.58e-3	1.79e-3	1.67	1.50e-3	1.96	97.49
32	1.04e-2	2.07	7.55e-3	4.93e-4	1.86	3.77e-4	1.98	19.59

TABLE 5. *A posteriori* error estimates using elliptic reconstruction and convergence rates under space refinement ; $T = 1$, $\tau = 0.1$

To assess the *a posteriori* error estimate using elliptic reconstruction, see (83), we evaluate the quantities

$$\eta_5 = \left(\sum_{m=1}^{N} \tau_m^{-1} (\widehat{\mathcal{E}}^m_{p,1}(\delta_t) + \widehat{\mathcal{E}}^m_u(\delta_t)) \right)^{\frac{1}{2}}, \qquad \eta_7 = (\widehat{\mathcal{E}}^N_u)^{\frac{1}{2}}, \qquad \eta_8 = (\widehat{\mathcal{E}}^N_{p,1})^{1/2}, \tag{86}$$

$$\eta_6 = \left(\tau_1 \|C\delta_t p_h^1 + B\delta_t u_h^1 - D_h p_{0h} - g_h^1\|_c^2 + \sum_{m=2}^{N} \tau_m^3 \|\delta_t (C\delta_t p_h^m + B\delta_t u_h^m - g_h^m)\|_c^2 \right)^{\frac{1}{2}}, \tag{87}$$

as well as the efficiency index

$$\mathcal{J}_{\text{eff}} = \frac{\eta_5 + \eta_6 + \eta_7 + \eta_8}{\|u^N - u^N_{h\tau}\|_a + \|p^N - p^N_{h\tau}\|_c}. \tag{88}$$

Recall that η_5 is associated with the space error indicator and that η_6 is associated with the time error indicator. Moreover, η_7 and η_8 control the error between the discrete solution and its elliptic reconstruction; see Lemma 4.1. Table 5 presents the results obtained under space refinement. The observed orders of convergence match theoretical predictions, with a slight super-convergence for η_5 and a slight sub-convergence for η_7 on the (very) coarse meshes. The quantity η_6, which is related to the time error, remains at a fairly constant value under space refinement. Additional tests (not reported here for brevity) indicate that η_6 converges with order close to 1 under time refinement. Finally, we observe that the efficiency index is quite large, especially on the (very) coarse meshes.

6. Conclusions

We have analyzed Euler–Galerkin approximations to coupled elliptic–parabolic problems with application to poroelasticity. In particular, we have obtained *a posteriori* error estimates delivering certified upper bounds for the approximation error in the sense that all the constants in the estimates have been specified. This can be an important feature in numerical simulations aiming at performance assessment of underground waste repositories. Furthermore, we have proposed an extension of the elliptic reconstruction technique introduced by Makridakis and Nochetto for linear parabolic problems to the present setting. This technique yields *a posteriori* error estimates that converge optimally for the u-component, but at the price of less favorable efficiency indices especially on (very) coarse meshes. Finally, we point out that the present work can be extended in many directions, including the use of time-dependent meshes and adaptive simulations driven by the present *a posteriori* error estimators.

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